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Weak separation axioms and weak covering properties

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ABSTRACT

We identify some remnants of normality and call them rudimentary normality, generalize the concept of submetacompact spaces to that of a weakly subparacompact space and that of a weakly* subparacompact space, and make a simultaneous generalization of collectionwise normality and screenability with the introduction of what is to be called collectionwise σ-normality. With these weak properties, we show that,
1) on weakly subparacompact spaces, countable compactness = compactness, ω_1-compactness = Lindelöfness;
2) on weakly subparacompact Hausdorff spaces with rudimentary normality, regularity = normality = countable paracompactness; and
3) on weakly subparacompact regular T_1-spaces with rudimentary normality, collectionwise σ-normality = screenability = collectionwise normality = paracompactness.
The famous Normal Moore Space Conjecture is thus given an even more striking appearance and Worrell and Wicke's factorization of paracompactness (over Hausdorff spaces) along with Krajewski's are combined and strengthened. The methodology extends itself to the factorization of paracompactness on locally compact, locally connected spaces in the manner of Gruenhage and on locally compact spaces in that of Tall, and to the factorization of subparacompactness and metacompactness in the genre of Katuta, Chaber, Junnila and Price and Smith and that of Boone, improving all of them.

Although separation axioms and covering properties come in from different directions, they are closely related and interact (see e.g. [1,3]) and they are best thought of as one.

We are to give, in Section 1 below, three (or four) weak covering or separation properties. We are to identify *some remnants of normality* that remain in the *de-normalization of collectionwise normality* of [10] and can be found within the *expandability* of Katetov and Krajewski [14] (although normality itself cannot) and call them *rudimentary normality* (a term suggested to me by Professor Brümmer); to give generalizations of the submetacompactness of Worrell and Wicke [24], to be called *weak and weak* subparacompactness*; and to make a simultaneous generalization of screenability and collectionwise normality, both of Bing [2], to be called *collectionwise σ-normality*, and a simultaneous generalization of *expandability*, strong screenability and collectionwise normality, to be called *strong collectionwise σ-normality*. The notion of collectionwise σ-*normality*, by its very nature, allows many variations leading to many different results.

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The results are:

1) on *weakly subparacompact* Hausdorff spaces with *rudimentary normality*, regularity = normality = countable paracompactness;

2) on *weakly subparacompact* spaces, *collectionwise σ-normality* $\Rightarrow$ screenability, while strong *collectionwise σ-normality* $\Rightarrow$ Krajewski's semiparacompactness;

3) *weak subparacompactness* + regularity + *rudimentary normality* + *collectionwise σ-normality* $\Rightarrow$ paracompactness;

4) on locally compact, locally connected Hausdorff spaces, *weak* subparacompactness* + *rudimentary normality* + $\mathfrak{c}$-*collectionwise ϵ-normality* $\Rightarrow$ paracompactness;

5) on locally compact spaces, *weak subparacompactness* + regularity + *rudimentary normality* + *collectionwise σ-normality with respect to compact subsets* $\Rightarrow$ paracompactness;

6) *weak subparacompactness* + countable metacompactness + *weak collectionwise σ-normality* = metacompactness;

7) *weak subparacompactness* + collectionwise δ^+-normality = subparacompactness;

8) on *weakly subparacompact spaces*, countable compactness = compactness, and *weakly subparacompact*, ω_1-compact spaces are Lindelöf;

9) an abstraction of Wicke and Worrell's result that, on weakly $\delta\theta$-refinable spaces, countable compactness = compactness.

0. Notations, terminology and preliminaries

0.1. No separation axioms are assumed. Thus, regular or normal spaces are not necessarily T_1. Paracompact (respectively, screenable and strongly screenable) spaces are those, every open cover of which has a locally finite (respectively, σ-disjoint and σ-discrete) open refinement, *at variance* with Michael's [15] usage of the term.

0.2. Undefined terms can be found defined in [4] and [12].

0.3. A topological space X is *normal* if (and only if) given disjoint closed subsets A and B of X, there is a sequence $\langle V_n\rangle$ of open subsets such that $B \subset \bigcup\{V_n\colon n \in \mathbb{N}\}$ and $\mathrm{Cl}\, V_n \subset X\backslash A$, for every $n \in \mathbb{N}$, (part of Theorem 3 of Hung [7]).

0.4. A topological space X is *countably paracompact* if, and only if, for a decreasing sequence $\langle F_n\rangle$ of closed sets with void intersection, there is an increasing sequence $\langle G_n\rangle$ of open sets such that $F_n \cap \mathrm{Cl}\, G_n = \emptyset$ and $\bigcup\{G_n\colon n \in \mathbb{N}\} = X$ (Ishikawa [11], Theorem V.6 of [17]).

0.5. Screenable, countably paracompact, normal spaces are strongly screenable (Nagami [16]).

0.6. Regular, strongly screenable spaces are paracompact (Michael [15]).

1. Three weak covering properties

We first identify a small part of *normality*, itself absent in the property of *regularity*, which, under some circumstances, is enough to make normal of the regular.

Definition 1.1. A topological space X is said to have *rudimentary normality* if, given a *discrete* family $\mathcal{C}$ of *closed* subsets and, for every $C \in \mathcal{C}$, given a neighborhood $U(C)$, disjoint from $C' \in \mathcal{C}$, when $C' \neq C$, there is, for every $n \in \mathbb{N}$, such a family $\mathcal{V}_n \equiv \{V_n(C)\colon C \in \mathcal{C}\}$ of open subsets that:

i) $C \subset \bigcup\{V_n(C)\colon n \in \mathbb{N}\} \subset U(C)$ for every $C \in \mathcal{C}$, and
ii) for every $n \in \mathbb{N}$, $\mathrm{Cl} \bigcup\{V_n(C)\colon C \in \mathcal{C}\} \subset \bigcup\{\mathrm{Cl}\, U(C)\colon C \in \mathcal{C}\}$.

Remarks. Clearly *normal* spaces have *rudimentary normality*. So does *the denormalization of collectionwise normality* (Definition 1.1 of [10]). In the property *expandability* of Katetov and Krajewski (see §4 of [12]), we can of course find *rudimentary normality*, although not normality itself. Clearly, ω_1-compact spaces have this property.

To exploit the newly defined property of *rudimentary normality*, we define a class, as big as possible, in which *regularity* under its influence becomes *normality*.

Definitions 1.2. Given a topological space X. A sequence $\{\mathcal{P}_\alpha\colon \alpha \in \omega^2\}$ of collections of subsets is said to be a *τ-sequence*, if,

i) $\mathcal{P}_{\omega m+n}$ restricted to any closed set Γ disjoint from $C_{\omega m+n} \equiv \bigcup\{\bigcup \mathcal{P}_\beta\colon \omega m \leqslant \beta < \omega m + n\}$ is *discrete and closed*, for all $0 \leqslant m, n < \omega$, and,
ii) $X = \bigcup\{\bigcup \mathcal{P}_\alpha\colon \alpha \in \omega^2\}$.

Given an open cover $\mathcal{U}$ of X, if there is a τ-sequence $\{\mathcal{P}_\alpha\colon \alpha \in \omega^2\}$ such that, for any $P \in \mathcal{P}_{\omega m+n}$ and any closed set Γ disjoint from $C_{\omega m+n}$, there is such a $U \in \mathcal{U}$ that $P \cap \Gamma \subset U$, we say $\mathcal{U}$ is *refined by the τ-sequence*. X is said to be *weakly subparacompact* if every open cover $\mathcal{U}$ of X is refined by some τ-sequence.

Remarks. 1. The *τ-sequence* can be shortened to one indexed by the ordinal ω, with the device that shows the union of countably many countable sets is countable. We decide to keep it as is so that its genesis from *submetacompactness* is more on the surface. Note that the families $\{\mathcal{P}_\alpha\colon \alpha \in [\omega 0, \omega 1)\}, \{\mathcal{P}_\alpha\colon \alpha \in [\omega 1, \omega 2)\}, \ldots$ are really independent, one of another, and they are strung together only for easy recall.

2. The concept *weak subparacompactness* thus defined is formally weaker than $B(D,\omega)$, the strongest among a host of properties, $B(D,\lambda)$, $B(LF,\lambda)$ and $B(HCP,\lambda)$, identified by J.C. Smith [20] for the purpose of exploring the area populated by *weak $\bar{\theta}$-refinability*, *weak θ-refinability*, *irreducibility* and the like. We are looking for a class biggest possible where we are, with rudimentary normality, allowed to scale the Separation Axioms from T_2 to T_4 and beyond.

Question. Is it true that weakly subparacompact spaces have property $B(D,\omega)$?

3. We can add a third item to the requirements in Definitions 1.2 thus,

iii) for every $P \in \bigcup\{\mathcal{P}_\alpha\colon \alpha \in \omega^2\}$, there is such a *finite* family $\mathcal{F}(P)$ of open neighborhoods of P that: a) $P \subset V \subset U$, for some $U \in \mathcal{U}$, if $V \in \mathcal{F}(P)$, b) for every $0 \leqslant m < \omega$, $p, q \in \omega$, $p < q$, $P \in \mathcal{P}_{\omega m+p}$, $Q \in \mathcal{P}_{\omega m+q}$; either $Q \subset \bigcup \mathcal{F}(P)$ or $Q \cap \bigcup \mathcal{F}(P) = \emptyset$,

and have instead *weak* subparacompactness*. Note that submetacompactness $\Rightarrow$ weak* subparacompactness $\Rightarrow$ weak subparacompactness.

4. Clearly, *subparacompact* spaces are weakly subparacompact, as are *submetacompact* spaces, when we note that, if, for some open cover $\mathcal{V}$ of X, we let $X_m \equiv \{x \in X\colon |\{V \in \mathcal{V}\colon x \in V\}| = m\}$, we see that $y \notin \operatorname{Cl} X_m$, if $m < |\{V \in \mathcal{V}\colon y \in V\}|$. *Weakly subparacompact* spaces, however, are not necessarily *submetacompact*. Counter-examples can be found in 4.5 and 4.9(i) of [4].

With these two newly defined properties, we are going to use them to strengthen Worrell and Wicke's and Krajewski's factorization of paracompactness and introduce below a simultaneous generalization of the two properties Bing introduced in his effort in the same direction.

Definition 1.3. A topological space X is said to have property *collectionwise σ-normality* (respectively, *strong* collectionwise σ-normality) if, given a *discrete* family $\mathcal{C}$ of *closed* subsets, there is, for every $n \in \mathbb{N}$, such a *disjoint* (respectively, *locally finite*) family $\mathcal{V}_n \equiv \{V_n(C)\colon C \in \mathcal{C}\}$ of open subsets that $C \subset \bigcup\{V_n(C)\colon n \in \mathbb{N}\}$, for every $C \in \mathcal{C}$.

Remarks. 1. The property collectionwise σ-normality is so named because, here, instead of the separating disjoint neighborhoods in the case of collectionwise normality, we have countably many families of disjoint open sets in order to separate in some manner the members of the discrete family of closed sets. Clearly, it is a simultaneous generalization of *screenability* and *collectionwise normality*, both of Bing [2]. It was first introduced in [8]. *Strong collectionwise σ-normality*, on the other hand, is a simultaneous generalization of *strong screenability* and collectionwise normality of Bing [2] and a property studied by Katetov and Krajewski (Theorem 4.1 of [12]). Variations of the notion of collectionwise σ-normality appear below in various places near where they are invoked, in the remarks 2) and 3) on Corollary 2.6 and in Definitions 2.7 and 3.1.

2. A sufficient condition for collectionwise σ-normality is that given a *discrete* family $\mathcal{C}$ of *closed* subsets, there is, for each $C \in \mathcal{C}$, such a (decreasing) sequence of open neighborhoods $\langle V_n(C)\rangle$ of C that, for every $\Gamma \in \mathcal{C}$, we have $\Gamma \cap \bigcap\{\operatorname{Cl} \bigcup\{V_n(C)\colon C \in \mathcal{C},\ C \neq \Gamma\}\colon n \in \mathbb{N}\} = \emptyset$.

2. Main results

The result 2.1 below is given first, not because it is the most important or the most profound but because it is the easiest to demonstrate. Its first part generalizes Worrell and Wicke [24] (Corollary 1.13 of [12]), though not Theorem 1.1 of Wicke and Worrell [23], which says essentially that, on a *countably compact* space, any open cover that can be arranged in the form of a *weak $\delta\theta$-sequence* (Definition analogous to that of θ-sequences) has a (countable and therefore) finite subcover. In Theorem 2.2 below, an abstraction of the possibility of such an arrangement is made.

Theorem 2.1. Weakly subparacompact*, countably compact spaces X are compact.* Weakly subparacompact*, ω_1-compact spaces are Lindelöf.*

Proof. Let $\mathcal{U}$ be any open cover of X refined by some τ-sequence $\{\mathcal{P}_\alpha\colon \alpha \in \omega^2\}$. We note that $\mathcal{P}_\alpha$ is *finite* if $\alpha = \omega m$, $0 \leqslant m < \omega$. It can be arranged that $P \subset U(P) \in \mathcal{U}$ for every $P \in \mathcal{P}_\alpha$. $\mathcal{U}_\alpha \equiv \{U(P)\colon P \in \mathcal{P}_\alpha\}$ is a *finite* subfamily of $\mathcal{U}$. Clearly, $\mathcal{P}_{\alpha+1}$, restricted to $X_1 \equiv X \backslash \bigcup \mathcal{U}_\alpha$, is *discrete and closed in* X and therefore also *finite*. It can be arranged that $P \cap X_1 \subset$

$U(P) \in \mathcal{U}$, for every $P \in \mathcal{P}_{\alpha+1}$ such that $P \cap X_1 \neq \emptyset$. $\mathcal{U}_{\alpha+1} \equiv \{U(P)\colon\ P \in \mathcal{P}_{\alpha+1},\ P \cap X_1 \neq \emptyset\}$ is a *finite* subfamily of $\mathcal{U}$. Clearly, $\mathcal{P}_{\alpha+2}$, restricted to $X_2 \equiv X_1 \backslash \bigcup \mathcal{U}_{\alpha+1}$, is *discrete and closed in* X and therefore *finite*. It can be arranged that $P \cap X_2 \subset U(P) \in \mathcal{U}$, for every $P \in \mathcal{P}_{\alpha+2}$ such that $P \cap X_2 \neq \emptyset \ldots$

Clearly $\mathcal{V} \equiv \bigcup \{\mathcal{U}_{\omega m+n}\colon\ 0 \leqslant m, n < \omega\}$ is a *countable* subfamily of $\mathcal{U}$ that covers X and there is a *finite* subcover $\mathcal{F} \subset \mathcal{V} \subset \mathcal{U}$ of X. X is therefore compact. □

Theorem 2.2. *A countably compact space X is compact, if (and only if), $(*)$ for every open cover $\mathcal{U}$ of X, there is such a countable cover $\mathcal{V} \equiv \{V_n\colon\ n \in \mathbb{N}\}$ that, for every $n \in \mathbb{N}$, there is an open neighborhood $\tilde{V}_n$ of V_n so that $C \subset V_n$ can be covered by a countable subfamily of $\mathcal{U}$, if $\operatorname{Cl} C \subset \tilde{V}_n$.*

Proof. Given $\mathcal{U}$ and $\mathcal{V}$, we define an increasing sequence $\{W_n\colon\ n \in \mathbb{N}\}$ of open subsets as follows. If V_1 can be covered by some countable subfamily $\mathcal{V}_1$, of $\mathcal{U}$, let $W_1 \equiv \bigcup \mathcal{V}_1$. Otherwise, let $W_1 = \tilde{V}_1$. If $V_2 \backslash W_1$ can be covered by some countable subfamily $\mathcal{V}_2$ of $\mathcal{U}$, let $W_2 \equiv W_1 \cup \bigcup \mathcal{V}_2$. Otherwise let $W_2 \equiv W_1 \cup \tilde{V}_2$, etc. Clearly, $\{W_n\colon\ n \in \mathbb{N}\}$ is an increasing open cover of X and $W_\nu = X$ for some $\nu \in \mathbb{N}$. Let μ be the largest natural number $\leqslant \nu$ such that $V_\mu \backslash W_{\mu-1} (W_0 \equiv \emptyset)$ cannot be covered by any countable subfamily of $\mathcal{U}$. If $\mu < \nu$, let $\mathcal{W} \equiv \bigcup \{\mathcal{V}_n\colon\ \mu < n \leqslant \nu\}$. If $\mu = \nu$, let $\mathcal{W} = \emptyset$. Either way $\mathcal{W}$ is (at most) *countable* and the set $X \backslash (W_{\mu-1} \cup \bigcup \mathcal{W}) \subset \tilde{V}_\mu$ is *closed*. It follows that $V_\mu \backslash (W_{\mu-1} \cup \bigcup \mathcal{W})$ and therefore $V_\mu \backslash W_{\mu-1}$ can be covered by some countable subfamily of $\mathcal{U}$, contradicting the definition of μ.

Therefore $W_n = W_{n-1} \cup \bigcup \mathcal{V}_n$ for $0 < n \leqslant \nu$, $X = \bigcup \{\bigcup \mathcal{V}_n\colon\ n \leqslant \nu\}$ and there is a finite $\mathcal{F} \subset \bigcup \{\mathcal{V}_n\colon\ n \leqslant \nu\}$ such that $X = \bigcup \mathcal{F}$, i.e., X is compact. □

Remarks. Clearly, if X is countably compact and weakly $\delta\theta$-refinable [23], we can identify the $\bigcup \mathcal{V}_n$'s of Wicke and Worrell with our $\tilde{V}_n$ and their $\{x \in X\colon\ 0 < \operatorname{ord}(\mathcal{V}_n, x) \leqslant \omega\}$ with our V_n and arrive at the conclusion of Theorem 1.1 of [23].

Corollary 2.3. *A countably compact space X is compact, if, for every open cover $\mathcal{U}$, there is a countable open cover $\{V_n\colon\ n \in \mathbb{N}\}$ such that every closed subset $C \subset V_n$, for some $n \in \mathbb{N}$, can be covered by a countable subfamily of $\mathcal{U}$.*

Theorem 2.4. *On* weakly subparacompact *Hausdorff spaces X with* rudimentary normality, *regularity = normality = countable paracompactness.*

Proof. 1) We are to prove that weakly subparacompact, countably paracompact Hausdorff spaces X, with rudimentary normality, are *regular*. Let W be an open neighborhood of $\xi \in X$. For every $x \notin W$, let $U(x)$ be such a neighborhood of x that $\xi \notin \operatorname{Cl} U(x)$. Let $\mathcal{U} \equiv \{U(x)\colon\ x \notin W\}$. The family $\mathcal{U} \cup \{W\}$ is an open cover of X, and is refined by some τ-sequence $\{\mathcal{P}_\alpha\colon\ \alpha \in \omega^2\}$. For all $\alpha \in \omega^2$, let $\mathcal{Q}_\alpha \equiv \{P \in \mathcal{P}_\alpha\colon\ P \backslash W \neq \emptyset\}$. We note $\mathcal{Q}_\alpha$ is a *discrete and closed* family in X, if $\alpha = \omega m$, $0 \leqslant m < \omega$. Choose $U(Q) \in \mathcal{U}$, so that $Q \subset U(Q)$, for every $Q \in \mathcal{Q}_\alpha$, and let $U'(Q) \equiv U(Q) \backslash \operatorname{Cl} \bigcup \{R\colon\ R \in \mathcal{Q}_\alpha,\ R \neq Q\}$. Because of *rudimentary normality* on X, there is, for every $n \in \mathbb{N}$ and every $Q \in \mathcal{Q}_\alpha$, an open subset $V_n(Q)$, as described in Definition 1.1, so that, for every $n \in \mathbb{N}$,

$$\begin{aligned}
\operatorname{Cl} \bigcup \bigl\{V_n(Q)\colon\ Q \in \mathcal{Q}_\alpha\bigr\} &\subset \bigcup \bigl\{\operatorname{Cl} U'(Q)\colon\ Q \in \mathcal{Q}_\alpha\bigr\} \\
&\subset \bigcup \bigl\{\operatorname{Cl} U(Q)\colon\ Q \in \mathcal{Q}_\alpha\bigr\} \\
&\subset X \backslash \{\xi\},
\end{aligned}$$

and if we write $V_{\alpha,n} \equiv \bigcup \{V_n(Q)\colon\ Q \in \mathcal{Q}_\alpha\}$, we can see that $\bigcup \mathcal{Q}_\alpha \subset V_\alpha \equiv \bigcup \{V_{\alpha,n}\colon\ n \in \mathbb{N}\}$. Clearly, $\mathcal{Q}_{\alpha+1}$, restricted to $X_1 \equiv X \backslash W \backslash V_\alpha$ is *discrete and closed in* X, and there is, for every $n \in \mathbb{N}$ and every $Q \in \mathcal{Q}_{\alpha+1}$, $Q \backslash V_\alpha \neq \emptyset$, an open subset $V_n(Q)$, there is, for every $n \in \mathbb{N}$, $V_{\alpha+1,n} \equiv \bigcup \{V_n(Q)\colon\ Q \in \mathcal{Q}_{\alpha+1},\ Q \backslash V_\alpha \neq \emptyset\}$, and there is $V_{\alpha+1} \equiv \bigcup \{V_{\alpha+1,n}\colon\ n \in \mathbb{N}\} \ldots$ Clearly, $X \backslash W \subset \bigcup \{V_{\alpha,n}\colon\ \alpha \in \omega^2,\ n \in \mathbb{N}\}$. Clearly, $\{V_{\alpha,n}\colon\ \alpha \in \omega^2,\ n \in \mathbb{N}\} \cup \{W\}$ is a *countable* open cover of X and there is a locally finite open refinement $\mathcal{G}$. Clearly, $X \backslash W \subset \bigcup \{G\colon\ G \in \mathcal{G},\ G \backslash W \neq \emptyset\} \subset \operatorname{Cl} \bigcup \{G\colon\ G \in \mathcal{G},\ G \backslash W \neq \emptyset\} \subset \bigcup \{\operatorname{Cl} V_{\alpha,n}\colon\ \alpha \in \omega^2,\ n \in \mathbb{N}\} \subset X \backslash \{\xi\}$. That is, X is regular.

2) We are to prove that weakly subparacompact regular (not necessarily Hausdorff) spaces X, with rudimentary normality, are *normal*. If, in 1) above, we let ξ be replaced by a closed subset A, we see that $A \cap \operatorname{Cl} V_{\alpha,n} = \emptyset$ for every $\alpha \in \omega^2$, $n = \mathbb{N}$, and in view of 0.3, we have normality, without recourse to countable paracompactness.

3) We are to prove that weakly subparacompact regular (not necessarily Hausdorff) spaces X, with rudimentary normality are *countably paracompact*. With 0.4 in mind, let there be a decreasing sequence $\langle F_n \rangle$ of closed sets with void intersection. For every point $x \in X$, let $U(x)$ be such that $F_n \cap \operatorname{Cl} U(x) = \emptyset$, where $n \in \mathbb{N}$ is the smallest possible. The family $\mathcal{U} \equiv \{U(x)\colon\ x \in X\}$ clearly covers X and is refined by some τ-sequence $\{\mathcal{P}_\alpha\colon\ \alpha \in \omega^2\}$. For each $\alpha \in \omega^2$, $\alpha = \omega m$, $0 \leqslant m < \omega$, and each $P \in \mathcal{P}_\alpha$, choose $U(P) \in \mathcal{U}$ so that $P \subset U(P)$ and $F_n \cap \operatorname{Cl} U(P) = \emptyset$ for some $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, let $\mathcal{P}_{\alpha,n} \equiv \{P\colon\ P \in \mathcal{P}_\alpha, F_n \cap \operatorname{Cl} U(P) = \emptyset\}$. Clearly, $\mathcal{P}_{\alpha,n}$ is *discrete and closed* and $\bigcup \{\mathcal{P}_{\alpha,n}\colon\ n \in \mathbb{N}\} = \mathcal{P}_\alpha$. For every $P \in \mathcal{P}_{\alpha,n}$, let $U'(P) \equiv U(P) \backslash \bigcup \{R \in \mathcal{P}_{\alpha,n}\colon\ R \neq P\}$. Because of *rudimentary normality* on X, there is, for every $l \in \mathbb{N}$ and every $P \in \mathcal{P}_{\alpha,n}$, an open subset $V_l(P)$, as described in Definition 1.1, so that, for every $l \in \mathbb{N}$,

$$\begin{aligned}\mathrm{Cl}\bigcup\{V_l(P):\ P\in\mathcal{P}_{\alpha,n}\} &\subset \bigcup\{\mathrm{Cl}\,U'(P):\ P\in\mathcal{P}_{\alpha,n}\}\\ &\subset \bigcup\{\mathrm{Cl}\,U(P):\ P\in\mathcal{P}_{\alpha,n}\}\\ &\subset X\backslash F_n,\end{aligned}$$

and if we write $V(\alpha,n,l)\equiv\bigcup\{V_l(P):\ P\in\mathcal{P}_{\alpha,n}\}$, we see that $F_n\cap\mathrm{Cl}\,V(\alpha,n,l)=\emptyset$. If we write $V(\alpha)\equiv\bigcup\{V(\alpha,n,l):\ n,l\in\mathbb{N}\}$, we see that $\mathcal{P}_{\alpha+1}$ restricted to $X\backslash V(\alpha)$ is *discrete and closed in* X, and there is, for every $n\in\mathbb{N}$, $\mathcal{P}_{\alpha+1,n}$, there is, for every $P\in\mathcal{P}_{\alpha+1,n}$ and every $l\in\mathbb{N}$, $V_l(P)$ and there is, for every $n,l\in\mathbb{N}$, $V(\alpha+1,n,l)$ so that $F_n\cap\mathrm{Cl}\,V(\alpha+1,n,l)=\emptyset\ldots.$

If, for every $n\in\mathbb{N}$, we let $G_n\equiv\bigcup\{V(\omega m+p,n,l):\ 0<l\leqslant n,\ 0\leqslant m,p\leqslant n\}$, we can see that $F_n\cap\mathrm{Cl}\,G_n=\emptyset$ and $\bigcup\{G_n:\ n\in\mathbb{N}\}=X$.

And, by 0.4, X is countably paracompact. □

Remarks. In view of the above, the famous Normal Moore Space Conjecture takes on an even more striking form: *Moore spaces with rudimentary normality* (1.1) *are metrizable*. Indeed, it was the realization of this possibility that prompted my looking for a biggest class of spaces possible to exploit the potential of the concept of rudimentary normality, the present paper being the result. While the properties of countable paracompactness and normality are in general distinct even among *perfect* spaces, they are indistinguishable among *weakly subparacompact* T_2 spaces with *rudimentary normality*. Thus the countably paracompact non-normal Moore spaces asserted to exist in Corollaries 2 and 3 of [22] actually fail to be normal at a fundamental level, in a specific area. Note also that pseudo-normal spaces do not always have *rudimentary normality* (see Example 3 of [22]) and T_1 Dowker Spaces cannot be *weakly subparacompact*. And a question of Younglove takes a seemingly more plausible form.

Question. Is it consistent that countably paracompact Moore spaces have *rudimentary normality* (cf. III, 5(a) of Tall [21])?

Theorem 2.5. Weakly subparacompact, collectionwise σ-normal *spaces X are screenable. Weakly subparacompact,* strongly *collectionwise σ-normal spaces X are semiparacompact* (*Krajewski's* [14] *terminology*).

Proof. Let $\mathcal{U}$ be an open cover of X that is refined by a τ-sequence $\{\mathcal{P}_\alpha:\ \alpha\in\omega^2\}$. $\mathcal{P}_\alpha$ being a *discrete and closed* collection, if $\alpha=\omega m$, $0\leqslant m<\omega$, there is, for every $n\in\mathbb{N}$ and every $P\in\mathcal{P}_\alpha$, an open set $V_n(P)$, as described in Definition 1.3, such that $\{V_n(P):\ P\in\mathcal{P}_\alpha\}$, for all $n\in\mathbb{N}$, is *disjoint* and $P\subset\bigcup\{V_n(P):\ n\in\mathbb{N}\}$. Choose $U(P)\in\mathcal{U}$ so that $P\subset U(P)$ and let $W_n(P)=U(P)\cap V_n(P)$, for every $P\in\mathcal{P}_\alpha$ and every $n\in\mathbb{N}$. Clearly, $\{W_n(P):\ P\in\mathcal{P}_\alpha\}$ is disjoint for all $n\in\mathbb{N}$. If we let $W_\alpha\equiv\bigcup\{W_n(P):\ P\in\mathcal{P}_\alpha,\ n\in\mathbb{N}\}$, we see that $\mathcal{P}_{\alpha+1}$ restricted to $X\backslash W_\alpha$ is *discrete and closed in* X, and there is, for every $n\in\mathbb{N}$ and every $P\in\mathcal{P}_{\alpha+1}$, $P\backslash W_\alpha\neq\emptyset$, an open set $V_n(P)$, as described in Definition 1.3, such that $\{V_n(P):\ P\in\mathcal{P}_{\alpha+1},\ P\backslash W_\alpha\neq\emptyset\}$, for all $n\in\mathbb{N}$, is *disjoint* and $P\backslash W_\alpha\subset\bigcup\{V_n(P):\ n\in\mathbb{N}\}$. Choose $U(P)\in\mathcal{U}$ so that $P\backslash W_\alpha\in U(P)$ and let $W_n(P)=U(P)\cap V_n(P)$, for all $P\in\mathcal{P}_{\alpha+1}$, $P\backslash W_\alpha\neq\emptyset$ and all $n\in\mathbb{N}$. If we let $W_{\alpha+1}\equiv W_\alpha\cup\bigcup\{W_n(P):\ P\in\mathcal{P}_{\alpha+1},\ P\backslash W_\alpha\neq\emptyset,\ n\in\mathbb{N}\}$.

$$\vdots$$

Clearly $\bigcup\{\{W_n(P):\ P\in\mathcal{P}_\alpha,\ P\backslash W_{\alpha-1}\neq\emptyset\ \text{if}\ \alpha\ \text{has an immediate predecessor}\}:\ n\in\mathbb{N},\ \alpha\in\omega^2\}$ is a σ-disjoint refinement of $\mathcal{U}$, i.e., X is *screenable*. The second statement is similarly proved. □

Remarks. The second statement strengthens Theorem 2.11 of Krajewski [14] (Theorem 4.1 of Junnila [12]). Note that *expandable spaces* are countably paracompact (Corollary 2.5.1 of [14]) and, in the presence of countable paracompactness, semiparacompactness = paracompactness.

Corollary 2.6. *Regular, weakly subparacompact, collectionwise σ-normal spaces with* rudimentary *normality are paracompact, i.e.,* on weakly subparacompact *regular spaces with* rudimentary normality, *collectionwise σ-normality = screenability = collectionwise normality = paracompactness. Regular, weakly subparacompact, strongly collectionwise σ-normal spaces are paracompact.*

Proof. Items 2 and 3 in the proof of Theorem 2.4 + Nagami (0.5) + Michael (0.6) + Theorem 2.5 yields the first result. Michael (0.6) + Theorem 2.5 yields the second. □

Remarks. 1. Clearly in the above, we have an improvement of both Worrell and Wicke (Theorem 4.16 of [4]) and Krajewski (Theorem 2.11 of [14]). One consequence is, when viewed with the factorization of *monotone developability* of [9] into three factors, we have the property of metrizability in *seven* factors, a large number of small pieces.

Question. It is well known [24] that on a submetacompact T_1-space, monotone developability = developability. On weakly subparacompact T_1-spaces?

2. If we define a *weak* form of collectionwise σ-normality by requiring $\mathcal{V}_n$ in Definition 1.3 to be only *point-finite*, we can say: *Countably metacompact*, weakly *subparacompact*, weakly *collectionwise σ-normal spaces are metacompact* (strengthening Boone, Theorem 3.3 of [12]).

3. If we restrict the $\mathcal{C}$ in Definition 1.3 to discrete families of *compact* closed subsets, we can speak of collectionwise σ-normality *with respect to compact subsets* in the manner Tall [21] does of collectionwise normality *with respect to compact subsets*, and say: On *locally compact, weakly subparacompact* regular spaces *with rudimentary normality, collectionwise σ-normality* with respect to compact subsets $\Rightarrow$ paracompactness (strengthening Tall, Theorem 1.7 of I of [21]).

The question of by how much the weakly subparacompact is short of being subparacompact arises immediately and naturally upon the definition of the former. By way of an answer we are to strengthen Junnila's *collectionwise δ-normality* in the direction towards subparacompactness itself. Thus,

Definition 2.7. A topological space is *collectionwise δ^+-normal*, if, for every discrete family $\mathcal{C}$ of closed subsets on X, there is a sequence $\langle \mathcal{V}_n \equiv \{V_n(C)\colon\ C \in \mathcal{C}\}\rangle$ of collections of open neighborhoods of members of $\mathcal{C}$, such that, for every $x \in X$, there is such a $\nu \in \mathbb{N}$ that $|\mathcal{V}_\nu(x) \equiv \{V \in \mathcal{V}_\nu\colon\ x \in V\}| \leqslant 1$ (cf. 3.7(ii) of [4]).

We can of course assume that, for every $n \in \mathbb{N}$, $C \in \mathcal{C}$,

i) $C \cap V_n(C') = \emptyset$, if $C' \neq C$, and
ii) $V_{n+1}(C) \subset V_n(C)$.

Note that, in particular, if $x \in \bigcap\{V_n(C)\colon\ n \in \mathbb{N}\}$, $|\mathcal{V}_\nu(x)| = 1$ for large enough ν's.

We give the theorem in the following without proof, it being so very straightforward once the idea is grasped.

Theorem 2.8. *(Cf. Katuta [13], Chaber [5], Theorem 2.7 of Junnila [12] and Price and Smith [18].) Weak subparacompactness + collectionwise δ^+-normality = subparacompactness.*

Remarks. 1. A pivot of the proof is provided below, in lieu of a full proof. Note that the $\mathcal{C}$ and the V's in Definition 2.7 beget, for every $n, k \in \mathbb{N}$, open sets $W_k(n) \supset X \setminus \bigcup\{V_n(C)\colon\ C \in \mathcal{C}\}$ and $E_k(n) \supset \bigcup \mathcal{C}$ such that, for every $x \in X$, $n \in \mathbb{N}$, there is such a $\kappa \in \mathbb{N}$ that $x \notin E_\kappa(n) \cap W_\kappa(n)$. If, for every $n, k \in \mathbb{N}$, $C \in \mathcal{C}$, we let

$$C(n,k) \equiv \bigl\{x \in X\colon\ x \in V_n(C),\ x \notin V_n\bigl(C'\bigr) \text{ if } C' \neq C,\ x \notin W_k(n)\bigr\},$$

we see that $\{C(n,k)\colon\ C \in \mathcal{C}\}$ for every $n, k \in \mathbb{N}$ is a *discrete closed* family and

$$\bigcap\bigl\{E_k(n)\colon\ n, k \in \mathbb{N}\bigr\} \subset \bigcup\bigl\{C(n,k)\colon\ C \in \mathcal{C},\ n, k \in \mathbb{N}\bigr\}.$$

2. It is not difficult to see that the difference between collectionwise δ^+-normality and collectionwise δ-normality can be accounted for with the notion of collectionwise ϵ-normality to be introduced in 3.1 below, provided we enlarge its range to include all discrete families of closed subsets (not merely the *compact* closed ones), a property clearly found in submetacompactness.

3. The question of the subparacompactness of submetacompact spaces has many solutions. Katuta [13] offered *subexpandability*, Chaber [5] *collectionwise sub-normality* and Junnila [12] *collectionwise δ-normality*. Chaber's collectionwise subnormality, equivalent to our collectionwise δ^+-normality, is in fact so strong that it allows the *submetacompactness* in Chaber's result to weaken to $B(D,\omega)$ (Theorem 3 of [18]), and here we further weaken it to *weak subparacompactness*.

3. Main results on locally compact, locally connected spaces

Given local compactness and local connectedness, normal Moore spaces are metrizable (Reed and Zenor [19]), normal, submetacompact spaces are paracompact (Gruenhage [6]). Thus collectionwise normality is dispensed with and simple normality suffices with the assumption of local compactness and local connectedness. We show in the following our weak covering properties and weak separation axioms can sharpen Gruenhage further. We introduce a simultaneous generalization of submetacompactness and the collectionwise δ^+-normality of Definition 2.7 above.

Definition 3.1. A topological space is *collectionwise ϵ-normal*, if, for every discrete family $\mathcal{C}$ of compact closed subsets on X, there is a sequence $\langle \mathcal{V}_n \equiv \{V_n(C)\colon\ C \in \mathcal{C}\}\rangle$ of collections of open neighborhoods of members of $\mathcal{C}$, such that, for every $x \in X$, there is such a $\nu \in \mathbb{N}$ that $|\mathcal{V}_\nu(x) \equiv \{V \in \mathcal{V}_\nu\colon\ x \in V\}| < \omega$.

We can of course assume that, for every $n \in \mathbb{N}$, $C \in \mathcal{C}$,

i) $C \cap V_n(C') = \emptyset$, if $C' \neq C$, and
ii) $V_{n+1}(C) \subset V_n(C)$.

If the C's are singletons, then we write $V_n(x)$ rather than $V_n(\{x\})$ and speak of the *collectionwise ϵ-Hausdorff* property. If the cardinality of the family $\mathcal{C}$ is always $\leqslant \mathfrak{c}$, then we attach the prefix $\mathfrak{c}$- to the word collectionwise.

Propositions 3.2 and 3.3 below demonstrate the interaction between these notions and that of *rudimentary* normality*, a stronger version of rudimentary normality, with $V_n(C) = V_1(C)$, in Definition 3.1, whatever C and whatever n.

Given, on a T_1 space X, a closed discrete subset A of cardinality $\mathfrak{c}$, there is a sequence $\langle \mathcal{P}_n \rangle$ of ever finer, finite, point-separating partitions of A. If X is regular and has *rudimentary* normality*, then, for every $x \in A$, there is a decreasing sequence $\langle U(x,n) \rangle$ of open neighborhoods such that

i) $\operatorname{Cl} U(x,0) \cap A = \{x\}$, and, for any $n > 0$,
ii) $\operatorname{Cl} \bigcup\{U(x,n)\colon x \in P\} \subset \bigcup\{\operatorname{Cl} U(x,n-1)\colon x \in P\}$, for all $P \in \mathcal{P}_n$.

Thus, for every $n > 0$, $\mathcal{P}_n$ can be separated by disjoint open subsets, i.e., every $P \in \mathcal{P}_n$ has an open neighborhood $U(P)$ such that $\{U(P)\colon P \in \mathcal{P}_n\}$ is a disjoint family of open sets. We can of course assume that $U(x,n) \subset U(P)$ if $x \in P \in \mathcal{P}_n$. If, further, X is locally connected and rim-compact, and $\mathfrak{c}$-collectionwise ϵ-Hausdorff, then we can assume that $\partial U(x,0)$ is compact for every $x \in A$ and all the $U(x,n)$'s are connected open neighborhoods and have inherited the properties of the $V_n(x)$'s described in Definition 3.1, and thus proceed to show that $(*)$ for every $x \in A$ there is an $n(x)$ such that $U(x,n(x)) \cap U(y,n(x)) = \emptyset$ for all $y \neq x$ (which implies that the members of A are separated by a family of disjoint open subsets). The negation of $(*)$ means that there are distinct $x, y_1, y_2, \ldots \in A$ such that $U(x,n) \cap U(y_n,n) \neq \emptyset$ whatever n and there is $z_n \in \partial U(x,0) \cap U(y_n,n)$ for every $n \in \omega$ clustering to $z \in \partial U(x,0)$. But then, there is a $\nu \in \omega$ such that $|\mathcal{V}_\nu(z)| < \omega$ and a $\mu \in \omega$ such that $x \in P \in \mathcal{P}_\mu$, $z, z_n \notin \operatorname{Cl} U(P)$, for some large enough n, and $y_n \notin P$, i.e., $U(x,\mu) \cap U(y_n,\mu) = \emptyset$, a contradiction. Therefore, we have,

Proposition 3.2. *On locally connected, rim-compact T_3 spaces,* $\mathfrak{c}$-collectionwise ϵ-Hausdorffness *with* rudimentary* normality $\Rightarrow$ *$\mathfrak{c}$-collectionwise Hausdorffness.*

And, *mutatis mutandis*, we have,

Proposition 3.3. *On locally connected, locally compact Hausdorff spaces X,* $\mathfrak{c}$-collectionwise ϵ-normality *with* rudimentary* normality $\Rightarrow$ *$\mathfrak{c}$-collectionwise normality with respect to compact subsets. Consequently, on X, the closure of any union of at most countably many open sets each with a compact closure cannot intersect every member of an uncountable discrete family of compact closed sets.*

The arguments advanced in the proof of Lemma 3 of [6] and that of Lemma 8.13 of [4] remaining good, *mutatis mutandis*, for the reduced circumstances of weak* subparacompactness (see third item of Remarks on Definition 1.2), we have,

Proposition 3.4. *If the spaces X of Proposition* 3.3 *above are in addition* connected *and* weakly* subparacompact, *then any cover with open sets each with a compact closure has a subcover of cardinality $\leqslant \omega_1$.*

Theorem 3.5. *On locally connected, locally compact Hausdorff spaces,* weak* subparacompactness $+$ $\mathfrak{c}$-collectionwise ϵ-normality $+$ rudimentary normality $\Rightarrow$ *paracompactness* (*cf. Corollary* 2.6 *above*).

Question. Is it true that *weak* subparacompactness* $+$ collectionwise ϵ-normality (with its range enlarged to include all discrete families of closed subsets) $\Rightarrow$ submetacompactness?

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